

# On the edge metric dimension for the random graph

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## Abstract

Let  $G(V, E)$  be a connected simple undirected graph. In this paper we prove that the edge metric dimension (introduced by Kelenc, Tratnik and Yero) of the Erdős-Rényi random graph  $G(n, p)$  is given by:

$$\text{edim}(G(n, p)) = (1 + o(1)) \frac{4 \log(n)}{\log(1/q)},$$

where  $q = 1 - 2p(1 - p)^2(2 - p)$ .

## 1 Introduction

In [1], Bollobás, Mitsche and Pralat computed an asymptotic for  $\dim(G(n, p))$  for a wide range of probabilities  $p(n)$  as a function of  $n$ . For instance, for constant  $p \in (0, 1)$ , it was shown that

$$\dim(G(n, p)) = (1 + o(1)) \frac{2 \log(n)}{\log(1/Q)},$$

where  $Q = p^2 + (1 - p)^2$ . In this paper we generalize the methods and calculations made by Bollobás, Mitsche and Pralat in [1] to give an asymptotic for  $\text{edim}(G(n, p))$ , which is a similar concept introduced by Kelenc, Tratnik and Yero in [2]. Namely, we show that

$$\text{edim}(G(n, p)) = (1 + o(1)) \frac{4 \log(n)}{\log(1/q)},$$

where  $q = 1 - 2p(1 - p)^2(2 - p)$ .

Metric dimension was introduced by Slater in [3] in 1975. The same idea was introduced independently by Harary and Melter in [4] a year later. Graphs with  $\dim(G) = 1$  and 2 were characterized in [5], and graphs with  $\dim(G) = n - 1$  and  $n - 2$  were described in [6]. This graph invariant is useful in areas like robot navigation ([5]), image processing ([7]), and chemistry ([6], [8], [9]).

The concept of edge metric dimension was introduced by Kelenc, Tratnik and Yero in [2] in 2016. They computed the edge metric dimension of a range of families of graphs and showed  $\text{edim}(G)$  can be less, equal to, or more than  $\dim(G)$ . They also showed computing  $\text{edim}(G)$  is NP-hard in general.

## 2 Definitions

Let  $G(V, E)$  be a finite, simple, undirected graph. For  $x, y \in E$ , we define the distance  $d(x, y)$  to be the length of the shortest path between  $x$  and  $y$ . For  $e = xy \in E$ ,  $v \in V$ , we define  $d(e, v) = \min\{d(x, v), d(y, v)\}$ .

Let  $R = \{r_1, \dots, r_{|R|}\} \subseteq V$ . Let  $d_R : V \cup E \rightarrow \mathbb{N}^{|R|}$  via  $(d_R(x))_i := d(x, r_i)$ . We say  $R$  distinguishes  $v_1, v_2 \in V$  if  $d_R(v_1) \neq d_R(v_2)$ , and similarly that  $R$  distinguishes  $e_1, e_2 \in E$  if  $d_R(e_1) \neq d_R(e_2)$ .

A set  $R \subseteq V$  is a *generating set* of a graph  $G(V, E)$  if, for any two vertices  $v_1$  and  $v_2$ ,  $d_R(v_1) \neq d_R(v_2)$ . A generating set with the smallest number of elements is called a *basis* of  $G$ , and the number of elements in a basis is the *dimension* of  $G$  (denoted  $\dim(G)$ ).

A set  $R \subseteq V$  is an *edge generating set* of a graph  $G(V, E)$  if for any two edges  $e_1$  and  $e_2$ ,  $d_R(e_1) \neq d_R(e_2)$ . An edge generating set with the smallest number of elements is called an *edge basis* of  $G$ , and the number of elements in the edge basis is the *edge dimension* of  $G$  (denoted  $\text{edim}(G)$ ).

We say  $f(n) = \mathcal{O}(g(n))$  if there exists a constant  $C > 0$  such that  $|f(n)| \leq C |g(n)|$ , and  $f(n) = o(g(n))$  if  $f = g \cdot o(1)$ , where  $o(1) \xrightarrow{n \rightarrow \infty} 0$ .

We will say that a property holds *asymptotically almost surely* (denoted a.a.s.) for the random graph if the probability that it holds for  $G(n, p)$  goes to 1 as  $n$  goes to infinity. We denote probability with  $\mathbb{P}$  and expected value with  $\mathbb{E}$ . In this paper we will compute the edge dimension of  $G(n, p)$ , where  $G(n, p)$  is the Erdős-Rényi random graph on  $n$  vertices with edge probability  $p$ , so that  $\mathbb{E}[|E|] = np$ .

### 3 The upper bound

In this section we prove the following theorem:

**Theorem 3.1.**

$$\text{edim}(G(n, p)) \leq (1 + o(1)) \frac{4 \log(n)}{\log(1/q)},$$

where  $q = 1 - 2p(1 - p)^2(2 - p)$ .

Before proving Theorem 3.1 we need some lemmas.

**Lemma 3.2.** *Fix  $\omega \in \{1, \dots, n\}$ . Suppose for any two distinct uniformly random edges  $e_1, e_2$  the following property holds: for a uniformly random subset  $W \subseteq V$  of size  $\omega$ ,*

$$\mathbb{P}(W \text{ does not distinguish } e_1, e_2) \leq 1/n^4 p^2.$$

*Then:*

$$\text{edim}(G) \leq \omega.$$

*Proof.* We use the probabilistic method. Indeed,

$$\mathbb{E}[|E(G(n, p))|] = p \binom{n}{2} < pn^2/2,$$

so the expected number of distinct pairs of edges is no more than  $\binom{pn^2/2}{2} \leq p^2 n^4 / 8$ . Then by our hypothesis the expected number of pairs not distinguished by some  $W \subseteq V$  with  $|W| = \omega$  is less than  $p^2 n^4 / 8 p^2 n^4 = 1/8$ . Since this is strictly less than 1, there must be at least one such set  $W$  that distinguishes all the pairs.  $\square$

**Lemma 3.3.** *In  $G(n, p)$ , the probability that a vertex  $v$  doesn't distinguish two uniformly random edges  $e_1, e_2$  is  $(1 + o(1))q$ , where  $q = 1 - 2p(1 - p)^2(2 - p)$ .*

*Proof.* There are two types of distinct edge pairs:

1.  $ab, bc$  for some  $a, b, c \in V$ .
2.  $ab, cd$  for  $a, b, c, d \in V$  and  $\{a, b\} \cap \{c, d\} = \emptyset$ .

Note that

$$\text{the expected number of type 2 pairs} = 3 \binom{n}{4} p^2 = \frac{n^4 p^2}{8} (1 + o(1)),$$

and

$$\text{the expected number of type 1 pairs} \leq n^3 = o\left(\frac{n^4 p^2}{8}\right).$$

Thus, we can neglect the type 1 pairs. Let  $xy, zt$  be a type 2 pair and  $v$  a uniformly random vertex. Clearly,  $\mathbb{P}(v \in \{x, y, z, t\}) = o\left(\frac{n^4 p^2}{8}\right)$ , so we can assume  $v$  is not a vertex of  $xy$  or  $zt$ . Since the random graph has diameter 2 a.a.s. (see [10]),  $v$  has distance 1 or 2 to  $x, y, z, t$  a.a.s.; moreover,  $\mathbb{P}(d(v, x) = 1) = p$ , so a.a.s.  $\mathbb{P}(d(v, x) = 2) = 1 - p$ . It is easy to see that  $v$  has distance 1 to  $xy$  and 2 to  $zt$  if and only if one of the following cases holds:

1.  $(d(v, x), d(v, y), d(v, z), d(v, t)) = (1, 1, 2, 2)$  (with probability  $p^2(1 - p)^2$ ).
2.  $(d(v, x), d(v, y), d(v, z), d(v, t)) = (1, 2, 2, 2)$  (with probability  $p(1 - p)^3$ ).
3.  $(d(v, x), d(v, y), d(v, z), d(v, t)) = (2, 1, 2, 2)$  (with probability  $p(1 - p)^3$ ).

The same probabilities hold for  $xy$  and  $zt$  switched. Thus, a.a.s.

$$\begin{aligned} \mathbb{P}(v \text{ distinguishes } xy, zt) &= (1 + o(1)) \cdot 2(p^2(1 - p)^2 + 2p(1 - p)^3) \\ &= (1 + o(1)) \cdot 2p(1 - p)^2(2 - p) = (1 + o(1))(1 - q). \end{aligned}$$

This gives us the desired result.  $\square$

**Lemma 3.4.** *Consider a uniformly random subset  $W \subseteq V(G(n, p))$  with*

$$|W| = (1 + o(1)) \frac{4 \log(n)}{\log(1/q)}.$$

*Then for uniformly random  $e_1$  and  $e_2 \in E(G(n, p))$ ,*

$$\mathbb{P}(W \text{ does not distinguish } e_1, e_2) \leq (1 + o(1))/n^4 p^2.$$

*Proof.* Using Lemma 3.3, we see that

$$\begin{aligned}
& \mathbb{P}(W \text{ doesn't distinguish } e_1, e_2) \\
& \leq (1 + o(1)) \mathbb{P}(\text{uniformly random vertex } v \text{ doesn't distinguish } e_1, e_2)^{|W|} \\
& \leq (1 + o(1)) q^{(1+o(1)) \frac{4 \log(n)}{\log(1/q)}} \\
& = (1 + o(1)) q^{-\log_q(n^4)} \\
& = (1 + o(1)) \frac{1}{n^4} \\
& \leq (1 + o(1)) \frac{1}{p^2 n^4}.
\end{aligned}$$

□

*Proof of Theorem 3.1.* Combining Lemma 3.4 and 3.2, we see that an edge basis of  $G(n, p)$  has cardinality at most

$$(1 + o(1)) \frac{4 \log(n)}{\log(1/q)},$$

which concludes the proof of Theorem 3.1. □

## 4 The lower bound

The goal of this section is to prove the following theorem:

**Theorem 4.1.** *For the random graph  $G(n, p)$  we have*

$$\text{edim}(G(n, p)) \geq (1 + o(1)) \frac{4 \log(n)}{\log(1/q)},$$

where  $q = 1 - 2p(1 - p)^2(2 - p)$ .

Let

$$\varepsilon := \frac{3 \log \log(n)}{\log(n)} = o(1).$$

We will show that a.a.s. there is no edge generating set  $R$  of cardinality less than

$$r := \frac{(4 - \varepsilon) \log(n)}{\log(1/q)}.$$

To do that we will use a theorem which is a version of Suen's inequality provided by Janson in [11]. First we introduce some notation:

- $\{I_i\}_{i \in \mathcal{I}}$  — a finite family of indicator random variables;
- $\Gamma$  — the associated dependency graph ( $\mathcal{I}$  is the set of vertices of  $\Gamma$ );
- For  $i, j \in \mathcal{I}$ , write  $i \sim j$  if  $i, j$  are adjacent in  $\Gamma$ ;
- $\mu := \sum_i \mathbb{P}(I_i = 1)$
- $\Delta := \sum_{i \sim j} \mathbb{E}[I_i I_j]$
- $\delta := \max_i \sum_{i \sim j} \mathbb{P}(I_j)$
- $S := \sum_i I_i$

**Theorem 4.2** (Suen's inequality: Theorem 2 of [11]).

$$\mathbb{P}(S = 0) \leq \exp(-\mu + \Delta \varepsilon^{2\delta})$$

We now apply this theorem to our problem.

Fix  $R \subseteq V(G(n, p))$  with  $|R| = r$ . Let

$$\mathcal{I} := \{(xy, zt) | xy, zt \in E(G(n, p)), xy \neq zt\}$$

be the set of pairs of pairs of distinct edges, and for any  $(xy, zt) \in \mathcal{I}$  let  $A_{xy,zt}$  to be the event  $d_R(xy) = d_R(zt)$  (with  $I_{xy,zt}$  being the corresponding indicator function). Let  $S = \sum_{(xy,zt) \in \mathcal{I}} I_{xy,zt}$ . Then

$$\mathbb{P}(R \text{ is an edge generating set}) = \mathbb{P}(S = 0).$$

The associated dependency graph has  $\mathcal{I}$  as vertices and  $(x_1 y_1, z_1 t_1) \sim (x_2 y_2, z_2 t_2)$  if and only if  $\{x_1, y_1, z_1, t_1\} \cap \{x_2, y_2, z_2, t_2\} \neq \emptyset$  (here, again,  $\sim$  denotes adjacency). Then by Theorem 4.2,

$$\mathbb{P}(S = 0) \leq \exp(-\mu + \Delta \varepsilon^{2\delta}), \tag{1}$$

where

$$\begin{aligned}\mu &:= \sum_{(e,f) \in \mathcal{I}} \mathbb{P}(A_{e,f}), \\ \Delta &:= \sum_{(e_1, f_1) \sim (e_2, f_2)} \mathbb{E}[I_{e_1 f_1} I_{e_2 f_2}], \\ \delta &:= \max_{(e_1, f_1) \in \mathcal{I}} \sum_{(e_2, f_2) \sim (e_1, f_1)} \mathbb{P}(A_{e_2, f_2}).\end{aligned}$$

We now show the following estimate for  $\mu$ :

**Lemma 4.3** (Evaluation of  $\mu$ ).

$$\mu = (1 + o(1))p^2 n^\varepsilon / 8.$$

*Proof.* Using Lemma 3.3, we can derive that that

$$\mathbb{P}(A_{e,f}) = (1 + o(1))q^r,$$

so, since the expected number of pairs is  $(1 + o(1))(n^4 p^2 / 8)$ , we indeed get

$$\mu = (1 + o(1))n^4 p^2 q^r / 8.$$

Since  $r = \frac{(4-\varepsilon)\log(n)}{\log(1/q)}$ ,

$$q^r = q^{-(4-\varepsilon)\log_q(n)} = n^{\varepsilon-4}. \quad (2)$$

Thus,

$$(1 + o(1))n^4 p^2 q^r / 8 = (1 + o(1))n^4 p^2 n^{\varepsilon-4} / 8 = (1 + o(1))p^2 n^\varepsilon / 8.$$

This means that, indeed,

$$\mu = (1 + o(1))p^2 n^\varepsilon / 8.$$

□

Now we estimate  $\Delta$  and show the following:

**Lemma 4.4** (Evaluation of  $\Delta$ ).

$$\Delta = o(\mu).$$

*Proof.*

**Claim.** *In calculating  $\Delta$ , we may only consider the adjacent pairs*

$$(x_1y_1, z_1t_1), (x_2y_2, z_2t_2) \in \mathcal{I}$$

for which

$$|\{x_1, y_1, z_1, t_1\} \cap \{x_2, y_2, z_2, t_2\}| = 1.$$

*Proof of claim.* Consider two adjacent elements of  $\mathcal{I} : (x_1y_1, z_1t_1) \sim (x_2y_2, z_2t_2)$ . Suppose  $|\{x_1, y_1, z_1, t_1, x_2, y_2, z_2, t_2\}| = 7$ . The expected number of such pairs is

$$p^4 \frac{n!}{16 \cdot (n-7)!} = (1 + o(1))p^4 n^7 / 16.$$

Now consider two adjacent elements of  $\mathcal{I}$  with  $|\{x_1, y_1, z_1, t_1, x_2, y_2, z_2, t_2\}| \leq 6$ . There are no more than

$$n^6 = o(p^4 n^7 / 16)$$

such pairs of pairs. □

Thus we can and will only consider pairs of elements of  $\mathcal{I}$  with only one vertex in common.

We will now compute the probability that  $I_{(x_1y_1, z_1t_1)} I_{(x_1y_2, z_2t_2)} = 1$ . Consider a uniformly random vertex  $v$ . We can neglect the case when  $v \in \{x_1, y_1, z_1, t_1, y_2, z_2, t_2\}$  because it happens with probability  $o(1)$ . Since the random graph has diameter 2 a.a.s.,  $I_{(x_1y_1, z_1t_1)} I_{(x_1y_2, z_2t_2)} = 1$  in the following cases:

Case 1:  $d_v(x_1) = 1$ . Then  $v$  has to have distance 1 to all four edges.  $v$  has distance 1 to  $z_1t_1$  (or  $z_2t_2$ ) with probability  $p^2 + 2p(1-p) = p(2-p)$ , and the distances from  $v$  to  $y_1, y_2$  don't affect anything, so

$$\mathbb{P}(I_{(x_1y_1, z_1t_1)} I_{(x_1y_2, z_2t_2)} = 1 \mid \text{case 1 holds}) = p^3(2-p)^2.$$

Case 2:  $d_v(x_1) = 2$ . Then  $v$  has distance 2 to both  $x_1y_1$  and  $z_1t_1$  with probability  $(1-p)^3$  and distance 1 to both  $x_1y_1$  and  $z_1t_1$  with probability  $p^2(2-p)$ . So  $v$  is equidistant from the two edges with probability  $(1-p)^3 + p^2(2-p)$ . Thus,

$$\mathbb{P}(I_{(x_1y_1, z_1t_1)} I_{(x_1y_2, z_2t_2)} = 1 \mid \text{case 2 holds}) = (1-p)((1-p)^3 + p^2(2-p))^2.$$



Hence the total probability

$$\mathbb{P}\left(I_{(x_1y_1,z_1t_1)}I_{(x_1y_2,z_2t_2)} = 1\right) = (1-p)((1-p)^3 + p^2(2-p))^2 + p^3(2-p)^2.$$

We will henceforth refer to this constant as  $s_p$ :

$$s_p := (1-p)((1-p)^3 + p^2(2-p))^2 + p^3(2-p)^2.$$

It follows that

$$\Delta = (1 + o(1))p^4n^7s_p^r/16.$$

Using (2), we get

$$\begin{aligned}\Delta &= (1 + o(1))p^4n^7s_p^r/16 \\ &= (1 + o(1))p^4n^3n^\varepsilon n^{4-\varepsilon}s_p^r/16 \\ &= (1 + o(1))\frac{p^2}{2}\left(\frac{s_p}{q}\right)^r\frac{p^2n^\varepsilon}{8} \\ &= (1 + o(1))\frac{p^2}{2}\left(\frac{s_p}{q}\right)^r\mu.\end{aligned}$$

Notice that

$$\begin{aligned}\left(\frac{s_p}{q}\right)^r &= \left(\frac{s_p}{q}\right)^{(4-\varepsilon)\log(n)/\log(q)} \\ &= n^{(4-\varepsilon)\log\left(\frac{s_p}{q}\right)/\log(1/q)} \\ &= n^{(4-\varepsilon)\left(\frac{-\log(s_p)}{\log(q)} - 1\right)} \\ &= n^{(4-\varepsilon)(-\log_q s_p - 1)} \leq n^{\varepsilon-4}.\end{aligned}$$

Thus,

$$(1 + o(1))\frac{p^2}{2}\left(\frac{s_p}{q}\right)^r\mu \leq (1 + o(1))\frac{p^2n^{\varepsilon-4}}{2}\mu = o(\mu).$$

This concludes the proof that

$$\Delta = o(\mu).$$

□

Finally, we estimate  $\delta$  and show the following:

**Lemma 4.5** (Evaluation of  $\delta$ ).

$$\delta = o(1).$$

*Proof.* Note that for fixed  $f_1, e_1$ ,

$$\begin{aligned} \mathbb{P}(A_{e_2, f_2} \mid (e_2, f_2) \text{ uniformly random}, (e_2, f_2) \sim (e_1, f_1)) = \\ \mathbb{P}(A_{e, f} \mid e, f \text{ uniformly random}). \end{aligned}$$

Thus, the maximum for  $\delta$  is achieved for  $(e_1, f_1)$  with the largest possible number of adjacent edge pairs  $(e_2, f_2)$ . Clearly, this number is the greatest when  $e_1$  and  $f_1$  don't share vertices. The expected number of adjacent edge pairs in this case is  $(1 + o(1))4p^2q^rn^3/2 = (1 + o(1))2n^3p^2$ . Since  $q^r = \mathbb{P}(A_{e, f})$  for uniformly random edges  $e, f$  we have

$$2\delta = (1 + o(1))2n^3p^2q^r.$$

Using (2), we get

$$\delta = (1 + o(1))2n^3p^2n^{\varepsilon-1} = o(1).$$

□

We are now ready to prove Theorem 4.1.

*Proof of Theorem 4.1.* Substituting the results of Lemma 4.3, 4.4, 4.5 into inequality (1), we obtain

$$\begin{aligned} \log(\mathbb{P}(S = 0)) &\leq (1 + o(1))(-\mu + o(\mu)e^{o(1)}) \\ &\leq (1 + o(1))(-\mu + o(\mu)) \\ &\leq -(1 + o(1))\mu \\ &\leq -(1 + o(1))p^2n^\varepsilon/8 \\ &\leq -p^2n^\varepsilon/16 \end{aligned}$$

for sufficiently large  $n$ . Then the expected number of edge generating sets of cardinality  $r$  is no greater than

$$\begin{aligned} \binom{n}{r} \exp(-p^2n^\varepsilon/16) &\leq n^r \exp(p^2n^\varepsilon/16) \\ &= \mathcal{O}(\exp[(4 - \varepsilon) \log^2(n) / \log(1/q) - p^2n^\varepsilon/16]) \\ &\leq \mathcal{O}(\exp[\log^2(n) - \log^3(n)p^2/16]) \\ &= o(1). \end{aligned}$$

This concludes the proof of Theorem 4.1.

□

## 5 Concluding remarks

We have shown that

$$\text{edim}(G(n, p)) = (1 + o(1)) \frac{4 \log(n)}{\log(1/q)},$$

where

$$q = 1 - 2p(1 - p)^2(2 - p).$$

As demonstrated by Bollobas in [1],

$$\dim(G(n, p)) = (1 + o(1)) \frac{2 \log(n)}{\log(1/Q)},$$

where  $Q = p^2 + (1 - p)^2$ . Since  $2/\log(1/Q) < 4/\log(1/q)$ , this means that

$$\dim(G(n, p)) < \text{edim}(G(n, p))$$

a.a.s. for all  $p \in (0, 1)$ .

While random graphs with constant edge probability don't help in resolving the problem of finding more examples of graphs  $G$  for which  $\text{edim}(G) < \dim(G)$  posed in [2], perhaps this problem could be addressed with random graphs of non-constant probability  $p(n)$ . Because of this it would be interesting to calculate  $\text{edim}(G(n, p(n)))$  for non-constant  $p(n)$ . As mentioned earlier, relevant results for  $\dim(G(n, p(n)))$  can be found in [1].

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